New perfect one-factorizations of complete graphs

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Abstract

We construct new perfect one-factorizations of the complete graphs K_{12168} and K_{29792}.

1 Introduction

A 1-factorization of the complete graph K_{2n} is a partition of the edge-set of K_{2n} into $2n-1$ 1-factors, each of which contains n edges that partition the vertex-set of K_{2n}. A perfect 1-factorization is a 1-factorization in which every pair of distinct 1-factors forms a Hamilton cycle.

It has been conjectured that a perfect 1-factorization of K_{2n} exists for all n. Two infinite classes of perfect 1-factorizations of K_{2n} are known to exist. When $2n = p+1$ (where p is prime), it is well-known that K_{2n} has a perfect 1-factorization. When $2n = 2p$ (where p is prime), Anderson [3] and Nakamura [13, 14] constructed perfect 1-factorizations of K_{2n}, independently, and it is known that these are isomorphic [8]. In addition to the two infinite families of perfect 1-factorizations, we only know of perfect 1-factorizations of several other orders. Perfect 1-factorizations of K_{16}, K_{28}, K_{244} and K_{344} were found by Anderson [1, 2, 4]. Perfect 1-factorizations of K_{36} were found by Seah and Stinson [16] and Kobayashi, Awoki, Nakazaki and Nakamura [9], independently. A perfect 1-factorization of K_{50} was found by Ihrig, Seah and Stinson [7]. A perfect 1-factorization of K_{40} was found by Seah and Stinson [17]. Perfect 1-factorizations of K_{1332} and K_{6860} were found by Kobayashi and Kiyasu [10]. Perfect 1-factorizations of $K_{126}, K_{170}, K_{730}, K_{1370}, K_{1850}, K_{2198}$ and K_{3126} were found by Dinitz and Stinson [6]. Most of these perfect 1-factorizations have been constructed by the method of starters. We note that the smallest order of K_{2n} for which a perfect 1-factorization is not known to exist is $2n = 52$.

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In this paper, we construct perfect 1-factorizations of K_{12168} and K_{29792} by the same method as the case $2n = 28, 244, 344, 1332, 6860$; these are the case $2n \equiv p^m + 1$, where p is prime and $p^m \equiv 3 \pmod 4$.

2 Semi-regular 1-factorizations

Let p be a prime number and m be a natural number such that $p^m \equiv 3 \pmod 4$. We put $q = p^m, s = (q-1)/2$ and $2n = q + 1$. $GF(q)$ denotes the Galois field with q elements. $K_{2n} = (V, E)$ denotes the complete graph on $2n$ vertices, and $V = GF(q) \cup \{\infty\}$, $E = \{\{x, y\} \mid x, y \in V, x \neq y\}$.

Let ω be a primitive element of $GF(q)$ and t be an odd integer such that $0 < t < 2s$. We define a starter 1-factor F_0:

$$F_0 = \{\{\omega^{2i}, \omega^{t+2i}\} \mid 0 \leq i \leq s-1\} \cup \{\{0, \infty\}\}.$$

For any $g \in GF(q)$,

$$\begin{aligned} F_g &= F_0 + g \\ &= \{\{\omega^{2i} + g, \omega^{t+2i} + g\} \mid 0 \leq i \leq s-1\} \cup \{\{g, \infty\}\}. \end{aligned}$$

is a 1-factor which is induced by the starter F_0. Then

$$F(\omega^t) = \{F_g \mid g \in GF(q)\}$$

is a 1-factorization of K_{2n}.

Let S_{2n} be the symmetric group of the set of all vertices of K_{2n}. Clearly if $\sigma \in S_{2n}$, then σ induces a permutation on the set of all edges of K_{2n} and on the set of all 1-factors of K_{2n}. We also use σ for these induced permutations. Now we define a symmetry group of a 1-factorization F:

$$\mathrm{Sym}(F) = \{\sigma \in S_{2n} \mid \sigma(F) = F\}.$$

Theorem 1. $\mathrm{Sym}(F(\omega^t))$ *is transitive and of order at least* $q(q-1)/2$.

Proof. For any $g \in GF(q)$, we define a permutation on V:

$$\tau_g(y) = \begin{cases} y + g & (y \in GF(q)) \\ \infty & (y = \infty). \end{cases}$$

It is easy to show that for any $g \in GF(q)$ and $y \in GF(q)$,

$$\tau_g(F_y) = F_{y+g},$$

hence

$$\tau_g \in \mathrm{Sym}(F(\omega^t)).$$

Since $\tau_g \neq \tau_{g'}$ $(g \neq g')$ and $\tau_{g+g'} = \tau_g\tau_{g'}$ $(g, g' \in GF(q))$, the additive group of $GF(q)$ is embedded in $\mathrm{Sym}(F(\omega^t))$, so $\mathrm{Sym}(F(\omega^t))$ has a subgroup of order q.

For any $g \in GF(q)^* = GF(q) \setminus \{0\}$, define a permutation σ_g on V:

$$\sigma_g(y) = \begin{cases} gy & (y \in GF(q)) \\ \infty & (y = \infty). \end{cases}$$

It is easy to show that for any $g \in GF(q)^{*2} = \{z^2 \mid z \in GF(q)^*\}$ and $y \in GF(q)$, we have

$$\sigma_g(F_y) = F_{gy},$$

hence

$$\sigma_g \in \mathrm{Sym}(F(\omega^t)).$$

Since $\sigma_g \neq \sigma_{g'}$ $(g \neq g')$ and $\sigma_{gg'} = \sigma_g\sigma_{g'}$ $(g, g' \in GF(q)^{*2})$, the multiplicative group of $GF(q)^{*2}$ is embedded in $\mathrm{Sym}(F(\omega^t))$, so $\mathrm{Sym}(F(\omega^t))$ has a cyclic subgroup of order $(q-1)/2$.

Thus q and $(q-1)/2$ divide the order of $\mathrm{Sym}(F(\omega^t))$, so $q(q-1)/2$ divides the order of $\mathrm{Sym}(F(\omega^t))$, since q and $(q-1)/2$ are relatively prime.

Take any two 1-factors F_g and F_h of $F(\omega^t)$. We have

$$\tau_{h-g}(F_g) = F_h,$$

so $\mathrm{Sym}(F(\omega^t))$ is transitive. □

Let F_g and F_h be 1-factors of K_{2n}. If

$$F_g \cup F_h = L_1 \cup L_2 \cup \ldots \cup L_k,$$

where L_i is a cycle with length l_i $(1 \leq i \leq k)$, and $l_1 + l_2 + \ldots + l_k = 2n$, we say the cycle structure of $F_g \cup F_h$ is of type $(l_1, l_2, \ldots, l_k)$. A 1-factorization F of K_{2n} is semi-regular if and only if for any 1-factors F_g, F_h, F_i, F_j in F $(g \neq h, i \neq j)$, the cycle structures of $F_g \cup F_h$ and $F_i \cup F_j$ are identical.

Theorem 2. [2, 4, 5, 12] *$F(\omega^t)$ is a semi-regular 1-factorization of K_{2n}.*

Proof. For any $g, h \in GF(q)$ $(g \neq h)$, we consider the cycle structure of $F_g \cup F_h$. Note that the cycle structure of $F_g \cup F_h$ is the same as the cycle structure of $\alpha(F_g \cup F_h)$, where $\alpha \in \mathrm{Sym}(F(\omega^t))$.

We know that either $(g-h) \in GF(q)^{*2}$ or $(h-g) \in GH(q)^{*2}$, since $-1 \notin GF(q)^{*2}$. We may now assume that $(h-g) \in GF(q)^{*2}$. We have

$$\tau_{-g}(F_g \cup F_h) = F_0 \cup F_{h-g}$$

and

$$\sigma_{(h-g)^{-1}}(F_0 \cup F_{h-g}) = F_0 \cup F_1.$$

Thus the cycle structure of $F_g \cup F_h$ is the same as the cycle structure of $F_0 \cup F_1$. Therefore $F(\omega^t)$ is semi-regular. □

3 Perfect 1-factorizations

A 1-factorization $F(\omega^t)$ is perfect if the cycle structure of $F_0 \cup F_1$ is of type $(2n)$. We apply this so as to find new perfect 1-factorizations. We checked perfectness of $F(\omega^1), F(\omega^3), F(\omega^5), \ldots$ until the first perfect 1-factorization was found. The search was stopped at the first success. SUN4/390 installed at University of Shizuoka was used. CPU time is written in parentheses. The results obtained are as follows.

In case $p = 11$ and $m = 3$, let ω be a primitive element of $GF(11^3)$ with a minimal polynomial $x^3 + x^2 + 5$. Then $F(\omega^7)$ is a perfect 1-factorization of K_{1332} (15 seconds) [10].

In case $p = 19$ and $m = 3$, let ω be a primitive element of $GF(19^3)$ with a minimal polynomial $x^3 + x^2 + 16$. Then $F(\omega^{127})$ is a perfect 1-factorization of K_{6860} (103 minutes) [10].

In case $p = 23$ and $m = 3$, let ω be a primitive element of $GF(23^3)$ with a minimal polynomial $x^3 + x^2 + 16$. Then $F(\omega^{19})$ is a perfect 1-factorization of K_{12168} (60 minutes).

In case $p = 31$ and $m = 3$, let ω be a primitive element of $GF(31^3)$ with a minimal polynomial $x^3 + x + 28$. Then $F(\omega^{373})$ is a perfect 1-factorization of K_{29792} (6424 minutes).

The last two perfect 1-factorizations are newly obtained.

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