

Another Construction of Dudeney Sets of K_{p+2}

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Abstract

Dudeney's round table problem was proposed about one hundred years ago. It is already solved when the number of people is even, but it is still unsettled except only few cases when the number of people is odd.

In this paper, another solution of Dudeney's round table problem is given when $n = p + 2$, where p is an odd prime number such that 2 or -2 is a primitive root of $GF(p)$. The method of constructing the solution is new.

1 Introduction

A Dudeney set in K_n is a set of Hamilton cycles with the property that every path of length two (2-path) in K_n lies on exactly one of the cycles. We call the problem of construction a Dudeney set in K_n for all natural numbers "Dudeney's round table problem".

A Dudeney set in K_n has been constructed when n is even [3]. In the case n is odd, a Dudeney set in K_n has been constructed only when

- (1) $n = 2^k + 1$ (k is a natural number) [4],
- (2) $n = p + 2$ (p is an odd prime number and 2 is a primitive root of $GF(p)$) [1],
- (3) $n = p + 2$ (p is an odd prime number and -2 is a primitive root of $GF(p)$) [2],

and some sporadic cases [1]. To construct a Dudeney set in K_n for a general odd integer $n > 0$, the case $n = p + 2$ (p is an odd prime number) plays an important role. But in the case, it has been constructed only when (2) and (3).

The method of constructing Dudeney sets in paper [2,3] is complex and it is difficult to extend to other cases. In this paper, we develop a new method of constructing Dudeney sets that is successful in both case (2) and case (3). The method is simple and elegant, so it is also possible to apply it to other cases with $n = p + 2$.

2 Preliminaries

Put $n_1 = p + 1$, where p is an odd prime number, and $r = (p - 1)/2$. We denote by $K_{n_1} = (V_{n_1}, E_{n_1})$ the complete graph on n_1 vertices, where $V_{n_1} = \{0, 1, 2, \dots, p-1\} \cup \{\infty\} = Z_p \cup \{\infty\}$ is the vertex set (Z_p is the set of integers modulo p).

For any integer i , $0 \leq i \leq p - 1$, define the 1-factors

$$F_i = \{\{\infty, i\}\} \cup \{\{a, b\} \in E_{n_1} \mid a, b \neq \infty, a + b \equiv 2i \pmod{p}\}$$

$$I_i = \{\{\infty, i/2\}\} \cup \{\{a, b\} \in E_{n_1} \mid a, b \neq \infty, a + b \equiv i \pmod{p}\}.$$

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Note that $F_i = 2I_i$, where multiplication is considered modulo p and we define $a \times \infty = \infty$ ($a \neq 0$).

Let σ be the vertex-permutation $(\infty)(0 \ 1 \ 2 \ \dots \ p-1)$, and put $\Sigma = \{\sigma^j \mid 0 \leq j \leq p-1\}$. When \mathcal{C} is a set of cycles or circuits in K_{n_1} , define $\Sigma\mathcal{C} = \{C^\tau \mid C \in \mathcal{C}, \tau \in \Sigma\}$.

For any edge $\{a, b\}$ in K_{n_1} , we define the length $d(a, b)$:

$$d(a, b) = \begin{cases} b - a \pmod{p} & (a, b \neq \infty) \\ \infty & (\text{otherwise}), \end{cases}$$

where we define that lengths c_1, c_2 are equal if $c_1 = c_2$ or $c_1 = -c_2 \pmod{p}$.

A set $H \subset Z_p^* = Z_p \setminus \{0\}$ is called a half-set modulo p if $|H| = (p-1)/2$ and $H \cup (-H) = Z_p^*$.

A sequence of non-zero integers $d = (d_1, d_2, \dots, d_t)$ is called a difference sequence of length t . Each component d_i is considered modulo p . We usually write d_i satisfying $-r \leq d_i \leq r$. For two difference sequences $d = (d_1, d_2, \dots, d_t)$ and $d' = (d'_1, d'_2, \dots, d'_t)$, we define $d = d'$ when $d_1 = d'_1, d_2 = d'_2, \dots, d_t = d'_t$ or $d_1 = -d'_t, d_2 = -d'_{t-1}, \dots, d_t = -d'_1$.

For an l -path $P = (a_0, a_1, \dots, a_l)$ ($a_i \neq \infty$ ($0 \leq i \leq l$)) in K_{n_1} , we define the difference sequence of P :

$$d(P) = (a_1 - a_0, a_2 - a_1, \dots, a_l - a_{l-1}).$$

Lemma 2.1 *Let P_1, P_2 be l -paths in K_{n_1} not containing ∞ . Then $d(P_1) = d(P_2)$ if and only if $P_2 = P_1^{\sigma^i}$ for some i , $0 \leq i \leq p-1$.*

We define the difference sequence of an Hamilton cycle in K_{n_1} as follows. Write a Hamilton cycle with ∞ the first. For a Hamilton cycle

$$C = (\infty, a_1, a_2, \dots, a_p),$$

define the difference sequence of C :

$$d(C) = (a_2 - a_1, a_3 - a_2, \dots, a_p - a_{p-1}).$$

Lemma 2.2 *Let C_1, C_2 be Hamilton cycles in K_{n_1} . Then $d(C_1) = d(C_2)$ if and only if $C_2 = C_1^{\sigma^i}$ for some i , $0 \leq i \leq p-1$.*

For a difference sequence $d = (a_1, a_2, \dots, a_{p-1})$ of length $p-1$, we call $W(d) = (\infty, 0, a_1, a_1 + a_2, \dots, \sum_{i=1}^{p-1} a_i)$ the representative Hamilton cycle of d , if $W(d)$ is a Hamilton cycle in K_{n_1} .

A difference sequence $d = (d_1, d_2, \dots, d_{p-1})$ of length $p-1$ is symmetric if $d_i = d_{p-i}$ ($1 \leq i \leq r$).

We next construct the complete graph K_n by adding a new vertex λ to K_{n_1} ; that is, put $n = n_1 + 1 = p + 2$, $K_n = (V_n, E_n)$ and $V_n = V_{n_1} \cup \{\lambda\}$. Extend σ to a permutation of V_n and denote it also by σ : $\sigma = (\infty)(\lambda)(0 \ 1 \ 2 \ 3 \ \dots \ p-1)$. Further we put $\Sigma = \{\sigma^j \mid 0 \leq j \leq p-1\}$.

Let A be a 1-factor in K_{n_1} which satisfies 1 and 2:

1. $F_0 \cup A$ is a Hamilton cycle in K_{n_1} .
2. If S is the multiset $\{d(a, b) \mid \{a, b\} \in A\}$, then we have $S = \{\infty, 1, 2, \dots, r\}$, i.e. A has all lengths.

If we insert the vertex λ into all the edges in A , we get a set of 2-paths in K_n . Denote this set by A^λ ; that is,

$$A^\lambda = \{(a, \lambda, b) \mid \{a, b\} \in A\}.$$

We note that paths are undirected, i.e., $(a, \lambda, b) = (b, \lambda, a)$. $F_0 \cup A^\lambda$ is considered to be a circuit in K_n .

Proposition 2.3 [5] *Assume h_i ($1 \leq i \leq r$) is a Hamilton cycle in K_{n_1} and $\Sigma\{h_i \mid 1 \leq i \leq r\}$ is a Dudeney set in K_{n_1} . Then*

$$\Sigma(\{F_0 \cup A^\lambda\} \cup \{h_i \mid 1 \leq i \leq r\})$$

has every 2-path in K_n exactly once.

3 Definition of $h(0)$

From now to the end of this paper, we assume that p is an odd prime number, $p \geq 19$, and that 2 or -2 is a primitive root of $GF(p)$.

Define a Hamilton cycle $h(0)$ in K_{n_1} as follows;

(i) When $p \equiv 1 \pmod{4}$,

$$h(0) = (\infty, 1, -1, -2, 2, 2^2, -2^2, -2^3, 2^3, \dots, -2^{r-1}, 2^{r-1}, 0).$$

(ii) When $p \equiv 3 \pmod{4}$,

$$h(0) = (\infty, 1, -1, -2, 2, 2^2, -2^2, -2^3, 2^3, \dots, 2^{r-1}, -2^{r-1}, 0).$$

As $h(0)$ contains the 1-factor F_0 , we can write $h(0) = F_0 \cup G$ where G is a 1-factor in K_{n_1} .

Lemma 3.1 *If S is the multiset $\{d(a, b) \mid \{a, b\} \in G\}$, then we have $S = \{\infty, 1, 2, \dots, r\}$, i.e. G has all lengths.*

4 Construction of $h(1)$

We would like to construct a Hamilton cycle $h(1)$ in K_{n_1} satisfying the following 2 conditions:

1. $\Sigma\{ah(1) \mid a \in H\}$ is a Dudeney set in K_{n_1} for any half-set H modulo p .
2. $h(1)$ has the 5-path $(\infty, 0, 1, -1, -2, 2)$.

We will construct $h(1)$ only when $p \equiv 3 \pmod{4}$. When $p \equiv 1 \pmod{4}$, we can construct $h(1)$ in a similar manner.

The difference sequence d of $F_0 \cup I_1$ is

$$d = (1, -2, 3, -4, 5, -6, 7, \dots, -(r-1), r; r, -(r-1), \dots, 7, -6, 5, -4, 3, -2, 1).$$

We transform it to make d_1 ;

$$d_1 = (1, -2, -1, 4, -5, 6, -7, \dots, (r-1), -r; -r, (r-1), \dots, -7, 6, -5, 4, -1, -2, 1).$$

The difference sequence $(-2, -1, 4)$ is in d_1 . If we multiply it by -3^{-1} , then we have $(2 \cdot 3^{-1}, 3^{-1}, -4 \cdot 3^{-1})$ which is in the first half of d_1 . Indeed, let b be an integer with $-r \leq b \leq r$ satisfying $b \equiv 3^{-1} \pmod{p}$, then b is even ($b = (p \pm 1)/3$) and we have $(2 \cdot 3^{-1}, 3^{-1}, -4 \cdot 3^{-1}) = (-(b-1), b, -(b+1))$, where the equality is considered as difference sequences.

Next, we transform d_1 to make d_2 : change $(-(b-1), b, -(b+1))$ to $(-(b-1), -1, (b+1))$ and change the sign from the next element to r , and then make d_2 symmetric, i.e.,

$$d_2 = (1, -2, -1, 4, -5, 6, -7, \dots, -(b-1), -1, (b+1), -(b+2), \dots, -(r-1), r; \\ r, -(r-1), \dots, -(b+2), (b+1), -1, -(b-1), \dots, -7, 6, -5, 4, -1, -2, 1).$$

Let $h(1)$ be the representative Hamilton cycle of d_2 , i.e., $h(1) = W(d_2)$. Then $h(1) = (\infty, 0, 1, -1, -2, 2, \dots)$ is a Hamilton cycle and it satisfies condition 2.

As $\Sigma\{a(F_0 \cup I_1) \mid a \in H\}$ (H is any half-set modulo p) is a Dudeney set in K_{n_1} , $\{ad \mid a \in H\}$ has the difference sequences of all 2-paths (a, b, c) in K_{n_1} with $a, b, c \neq \infty$ exactly once. Since

$$-3^{-1}(-2, -1, 4) = (-(b-1), b, -(b+1))$$

$$-3^{-1}(-2, 3, -4) = (-(b-1), -1, (b+1)) \text{ or } -(-(b-1), -1, (b+1)),$$

all difference sequences in $\{ad \mid a \in H\}$ of length 2 and all difference sequences in $\{ad_2 \mid a \in H\}$ of length 2 are the same. Therefore, we have Prop. 4.1 which shows that $h(1)$ satisfies condition 1.

Proposition 4.1 *For any half-set H modulo p , $\{ad_2 \mid a \in H\}$ has the difference sequences of all 2-paths (a, b, c) in K_{n_1} with $a, b, c \neq \infty$ exactly once.*

5 Construction of a Dudeney set

Let G be the 1-factor defined in §3. Insert the vertex λ into all edges in G and define G^λ same as before; that is,

$$G^\lambda = \{(a, \lambda, b) \mid \{a, b\} \in G\}.$$

Put $h(a) = ah(1)$, where a is an integer $\neq 0$. Since G has all lengths (Lemma 3.1), we obtain by Prop. 2.3,

Proposition 5.1 *Let H be a half-set modulo p . Then*

$$\Sigma(\{F_0 \cup G^\lambda\} \cup \{h(a) \mid a \in H\})$$

has every 2-path in K_n exactly once.

We leave one λ in $F_0 \cup G^\lambda$ and scatter the remaining r λ s over $\{h(a) \mid a \in H\}$ to construct a Dudeney set in K_n .

Put $H_1 = \{(-2)^i \mid 0 \leq i \leq r-2\}$. As $h(1)$ has a 3-path $(1, -1, -2, 2)$, $h((-2)^i)$ has a 3-path $((-2)^i, -(-2)^i, -2(-2)^i, 2(-2)^i)$. Denote $h((-2)^i)^\lambda$ the cycle in K_n obtained from $h((-2)^i)$ inserting λ into the center of this 3-path, i.e., inserting λ between $-(-2)^i$ and $-2(-2)^i$.

$h(1)$ has a 3-path $(\infty, 0, 1, -1)$. If G has $\{r, 0\}$, put $H_0 = H_1 \cup \{r\}$ and denote $h(r)^\lambda$ the cycle obtained from $h(r)$ inserting λ into the center of the 3-path $(\infty, 0, r, -r)$. If G has

$\{-r, 0\}$, put $H_0 = H_1 \cup \{-r\}$ and denote $h(-r)^\lambda$ the cycle obtained from $h(-r)$ inserting λ into the center of the 3-path $(\infty, 0, -r, r)$.

Denote $h(0)^\lambda$ the cycle obtained from $h(0)$ inserting λ into the center of $\{\infty, 1\}$. Then we have the following Proposition which is our main theorem.

Proposition 5.2

$$\mathcal{D} = \Sigma(\{h(0)^\lambda\} \cup \{h(a)^\lambda \mid a \in H_0\})$$

is a Dudeney set in K_n .

6 Example

Put $p = 19$, then $n = 21$, $r = 9$ and 2 is a primitive root of $GF(p)$. We have

$$h(0) = (\infty, 1, -1, -2, 2, 4, -4, -8, 8, -3, 3, 6, -6, 7, -7, 5, -5, 9, -9, 0)$$

$$d = (1, -2, 3, -4, 5, -6, 7, -8, 9, 9, -8, 7, -6, 5, -4, 3, -2, 1)$$

$$d_1 = (1, -2, -1, 4, -5, 6, -7, 8, -9, -9, 8, -7, 6, -5, 4, -1, -2, 1)$$

$$d_2 = (1, -2, -1, 4, -5, -1, 7, -8, 9, 9, -8, 7, -1, -5, 4, -1, -2, 1)$$

$$h(1) = (\infty, 0, 1, -1, -2, 2, -3, -4, 3, -5, 4, -6, 5, -7, -8, 6, -9, 9, 7, 8).$$

Put $H_0 = H_1 \cup \{-r\} = \{1, -2, 4, -8, -3, 6, 7, 5, -9\}$. The following $r + 1$ cycles and their rotations by Σ make a Dudeney set in K_n .

$$h(0)^\lambda = (\infty, \lambda, 1, -1, -2, 2, 4, -4, -8, 8, -3, 3, 6, -6, 7, -7, 5, -5, 9, -9, 0)$$

$$h(1)^\lambda = (\infty, 0, 1, -1, \lambda, -2, 2, -3, -4, 3, -5, 4, -6, 5, -7, -8, 6, -9, 9, 7, 8)$$

$$h(-2)^\lambda = (\infty, 0, -2, 2, \lambda, 4, -4, \dots)$$

$$h(4)^\lambda = (\infty, 0, 4, -4, \lambda, -8, 8, \dots)$$

$$h(-8)^\lambda = (\infty, 0, -8, 8, \lambda, -3, 3, \dots)$$

$$h(-3)^\lambda = (\infty, 0, -3, 3, \lambda, 6, -6, \dots)$$

$$h(6)^\lambda = (\infty, 0, 6, -6, \lambda, 7, -7, \dots)$$

$$h(7)^\lambda = (\infty, 0, 7, -7, \lambda, 5, -5, \dots)$$

$$h(5)^\lambda = (\infty, 0, 5, -5, \lambda, 9, -9, \dots)$$

$$h(-9)^\lambda = (\infty, 0, \lambda, -9, 9, -1, 1, \dots).$$

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