

# Perfect 1-Factorizations of Complete Bipartite Graphs

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**Abstract** In this paper, we prove that there exists a perfect 1-factorization of the complete bipartite graph  $K_{n,n}$  if there exists a perfect 1-factorization of the complete graph  $K_{n+1}$ , and when  $n > 2$  is even, there does not exist perfect 1-factorizations of  $K_{n,n}$ .

## 1 Introduction

Let  $K_n$  be the complete graph on  $n$  vertices and  $K_{n,n}$  the complete bipartite graph on vertex-set  $\{1, 2, \dots, n\} \cup \{1', 2', \dots, n'\}$ .

A 1-factorization  $\mathcal{F}$  of a graph  $G$  is called perfect if for any two distinct 1-factors  $F_1, F_2 \in \mathcal{F}$ ,  $F_1 \cup F_2$  is a hamilton cycle of  $G$ .

It has been conjectured that a perfect 1-factorization of  $K_n$  exists for all even number  $n(> 2)$ . Two infinite classes of perfect 1-factorizations of  $K_n$  are known to exist. When  $n = p + 1$  (where  $p$  is prime), it is well-known that  $K_n$  has a perfect 1-factorization. When  $n = 2p$  (where  $p$  is prime), Anderson and Nakamura constructed perfect 1-factorizations of  $K_n$  independently, and it is known that they are isomorphic. In addition to the two infinite families of perfect 1-factorizations, we only know of perfect 1-factorizations of several other orders:  $n = 16, 28, 36, 40, 50, 126, 170, 244, 344, 730, 1332, 1370, 1850, 2198, 3126, 6860, 12168, 29792$  [4].

In this paper we will show that there exists a perfect 1-factorization of the complete bipartite graph  $K_{n,n}$  if there exists a perfect 1-factorization of the complete graph  $K_{n+1}$ , and when  $n > 2$  is even, there does not exist perfect 1-factorizations of  $K_{n,n}$ .

## 2 Perfect 1-factorizations of $K_{n+1}$ and $K_{n,n}$

Let  $K_{n+1}$  be the complete graph with vertex-set  $\{0, 1, 2, \dots, n\}$  and  $K_{n,n}$  the complete bipartite graph with vertex-set  $\{1, 2, \dots, n\} \cup \{1', 2', \dots, n'\}$ . Let  $\mathcal{F}$  be a perfect 1-factorization of  $K_{n+1}$  if there exists. By the assumption,  $n$  is

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odd. For a 1-factor  $F = \{\{0, c\}, \{a_1, b_1\}, \{a_2, b_2\}, \dots, \{a_{(n-1)/2}, b_{(n-1)/2}\}\} \in \mathcal{F}$ , we define a 1-factor  $F'$  of  $K_{n,n}$ :

$$F' = \{\{c, c'\}, \{a_1, b'_1\}, \{a'_1, b_1\}, \{a_2, b'_2\}, \{a'_2, b_2\}, \dots, \{a_{(n-1)/2}, b'_{(n-1)/2}\}, \{a'_{(n-1)/2}, b_{(n-1)/2}\}\}.$$

Put  $\mathcal{F}' = \{F' \mid F \in \mathcal{F}\}$ , then it is clear that  $\mathcal{F}'$  is a 1-factorization of  $K_{n,n}$ .

Let  $F_1$  and  $F_2$  be any two 1-factors of  $\mathcal{F}$  and let  $(0, c_1, c_2, \dots, c_n)$  be the hamilton cycle of  $F_1 \cup F_2$ , where  $\{0, c_1\} \in F_1, \{c_1, c_2\} \in F_2, \dots, \{c_n, 0\} \in F_2$ . Then we have

$$F'_1 \cup F'_2 = (c_1, c'_1, c_2, c'_2, c_3, c'_3, \dots, c'_n, c_n, c'_{n-1}, c_{n-2}, \dots, c'_4, c_3, c'_2),$$

which shows  $F'_1 \cup F'_2$  is a hamilton cycle. Thus  $\mathcal{F}'$  is perfect and we have the following theorem.

**2.1 Theorem.** *There exists a perfect 1-factorization of the complete bipartite graph  $K_{n,n}$  if there exists a perfect 1-factorization of the complete graph  $K_{n+1}$ , where  $n$  is an odd integer  $> 1$ .*

### 3 Perfect 1-factorizations of $K_{n,n}$ when $n$ is even

Let  $K_{n,n}$  be the complete bipartite graph with vertex-set  $\{1, 2, \dots, n\} \cup \{1', 2', \dots, n'\}$ . Let  $F_1$  be a 1-factor of  $K_{n,n}$ :

$$F_1 = \{\{1, 1'\}, \{2, 2'\}, \dots, \{n, n'\}\}.$$

Let  $F = \{\{1, i'_1\}, \{2, i'_2\}, \dots, \{n, i'_n\}\}$  be any 1-factor such that  $F \cap F_1 = \phi$ , and define  $\sigma_F \in S_n$ :

$$\sigma_F = \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix},$$

where  $S_n$  is the symmetric group on  $\{1, 2, \dots, n\}$ .

**3.1 Lemma.** *Put  $F_1 = \{\{1, 1'\}, \{2, 2'\}, \dots, \{n, n'\}\}$  and let  $F = \{\{1, i'_1\}, \{2, i'_2\}, \dots, \{n, i'_n\}\}$  be a 1-factor of  $K_{n,n}$  such that  $F \cap F_1 = \phi$ . The following are equivalent.*

- (i)  $F \cup F_1$  is a hamilton cycle of  $K_{n,n}$ .
- (ii)  $\sigma_F$  is a cycle of length  $n$ .

*Proof.* Let

$$\sigma_F = (1 \ j_1 \ j_2 \ \dots \ j_s)(k_1 \ k_2 \ \dots \ k_t) \dots$$

be the cycle decomposition of  $\sigma_F$ . Then  $F \cup F_1$  contains a cycle

$$1 \rightarrow j'_1 \rightarrow j_1 \rightarrow j'_2 \rightarrow j_2 \rightarrow \dots \rightarrow j'_s \rightarrow j_s \rightarrow 1' \rightarrow 1$$

which Lemma 3.1 follows.  $\square$

**3.2 Lemma.** *Let  $F$  and  $F'$  be 1-factors of  $K_{n,n}$  such that  $F \cap F_1 = \phi$  and  $F' \cap F_1 = \phi$ . The following are equivalent.*

- (i)  $F \cup F'$  is a hamilton cycle of  $K_{n,n}$ .
- (ii)  $\sigma_{F'}^{-1} \sigma_F$  is a cycle of length  $n$ .

*Proof.* Let  $F = \{\{1, i'_1\}, \{2, i'_2\}, \dots, \{n, i'_n\}\}$  and  $F' = \{\{1, j'_1\}, \{2, j'_2\}, \dots, \{n, j'_n\}\}$ , then

$$\sigma_F = \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix},$$

$$\text{and } \sigma_{F'} = \begin{pmatrix} 1 & 2 & \dots & n \\ j_1 & j_2 & \dots & j_n \end{pmatrix}.$$

$F \cup F'$  contains a cycle

$$\begin{aligned} 1 \rightarrow \{\sigma_F(1)\}' = i'_1 &\rightarrow \sigma_{F'}^{-1}(\sigma_F(1)) \rightarrow \{\sigma_F(\sigma_{F'}^{-1}(\sigma_F(1)))\}' \\ &\rightarrow \sigma_{F'}^{-1}(\sigma_F(\sigma_{F'}^{-1}(\sigma_F(1)))) \rightarrow \dots \end{aligned}$$

which Lemma 3.2 follows.  $\square$

**3.3 Theorem.** *If  $n > 2$  is even, there does not exist perfect 1-factorizations of the complete bipartite graph  $K_{n,n}$ .*

*Proof.* Suppose there exists a perfect 1-factorization  $\mathcal{F}$  of  $K_{n,n}$ , where  $n > 2$  is even. We can assume  $F_1 = \{\{1, 1'\}, \{2, 2'\}, \dots, \{n, n'\}\} \in \mathcal{F}$  without loss of generality. We have  $|\mathcal{F}| \geq 4$  as  $n > 2$ . Let  $F$  and  $F'$  be any two 1-factors in  $\mathcal{F}$  such that  $F, F' \neq F_1$ . Since  $F \cup F_1$  and  $F' \cup F_1$  are hamilton cycles,  $\sigma_F$  and  $\sigma_{F'}$  are cycles of length  $n$  from Lemma 3.1. Therefore  $\sigma_F$  and  $\sigma_{F'}$  are odd permutations as  $n$  is even. Since  $F \cup F'$  is a hamilton cycle,  $\sigma_{F'}^{-1} \sigma_F$  is a cycle of length  $n$  from Lemma 3.2. Thus  $\sigma_{F'}^{-1} \sigma_F$  is also an odd permutation which gives a contradiction.  $\square$

## References

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